Complex Numbers

$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ where $i^2 = -1$.

Note $\mathbb{C}$ has two operations:

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i$$

$$(a_1 + b_1 i) \cdot (a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i^2$$
$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_2 a_1) i$$

Observation: when $b_1 = 0$, $a_1 (a_2 + b_2 i) = (a_1 a_2) + (a_1 b_2) i$

The Complex numbers form a (real) vector space!

Even better: Use complex numbers instead of real numbers when defining vector spaces...

This yields Complex vector spaces!

Ex: $\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : a, b, c \in \mathbb{C} \right\} = \mathbb{C}^3$.

NB: Everything we've done so far can be extended to complex vector spaces as well ☺.

Point: Don't be afraid of complex numbers...

Last Time: The eigenvalues of a matrix M are the roots of the characteristic polynomial $P_M(\lambda)$.

$$P_M(\lambda) = \det(M - \lambda I)$$

Ex: Compute E-values of $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Sol: $P_M(\lambda) = \det\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

$$= \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)^2 - (1 \cdot -1) = (1-\lambda)^2 + 1$$

$\therefore \; P_M(\lambda) = 0 \iff (1-\lambda)^2 + 1 = 0$

$$\iff (1-\lambda)^2 = -1$$

$$\iff 1-\lambda = \pm i$$

$$\iff \lambda = 1 \pm i$$

$\therefore$ M has complex eigenvalues! ▨

Q: Given one E-value, what are it's eigenvectors?

Ex: Consider $M = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$.

$$P_M(\lambda) = \det \begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}$$

$$= (3-\lambda)(2-\lambda) - 2$$

$$= 6 - 5\lambda + \lambda^2 - 2$$

$$= \lambda^2 - 5\lambda + 4$$

$$= (\lambda - 4)(\lambda - 1)$$

$\therefore \; P_M(\lambda) = 0 \iff \lambda = 4$ or $\lambda = 1$.

Because E-vectors must satisfy

$$Mv = \lambda v \quad \text{i.e. } (M-\lambda I)v = 0$$

$$\text{i.e. } v \in \text{null}(M-\lambda I),$$

we can find E-vectors by computing null$(M-\lambda I)$!

For $\lambda = 4$: $M - \lambda I = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$

$$\therefore \begin{array}{cc} x & y \\ \end{array} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right] \rightsquigarrow \begin{array}{cc} x & y \\ \end{array} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

← Solving $(M-4I)\begin{bmatrix} x \\ y \end{bmatrix} = 0$

when $\{-x + y = 0$ we have solution!

Point: $\begin{bmatrix} x \\ x \end{bmatrix}$ should be an eigenvector for $\lambda = 4$

Check: $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 4x \\ 4x \end{bmatrix} = 4\begin{bmatrix} x \\ x \end{bmatrix}$ ✓

$\therefore \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is a basis of eigenspace of $\lambda = 4$.

(Recall: Eigenspace associated to λ is
$V_\lambda := \{v \in V : Mv = \lambda v\}$).

For $\lambda = 1$: Compute null$(M - 1I)$

$$M - I = \begin{bmatrix} 3-1 & 1 \\ 2 & 2-1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{cases} 2x + y = 0 \\ 0 = 0 \end{cases} \rightsquigarrow y = -2x$$

Thus $\left\{\begin{bmatrix}1\\-2\end{bmatrix}\right\}$ forms a basis for E-space V_1.

<u>Check:</u> $M\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}3&1\\2&2\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}3-2\\2-4\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix} = 1\begin{bmatrix}1\\-2\end{bmatrix}$ ✓

Hence, we have $B = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}\right\}$ a basis of eigenvectors of M for $\mathbb{R}^2$...

On a whim: Let's compute $\operatorname{Rep}_{B,B}(L_M)$.

where $\operatorname{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(L_M) = M$:

$$\ast\ \operatorname{Rep}_{\mathcal{E}_2,B}(id)^{-1} = \begin{bmatrix}1&1\\1&-2\end{bmatrix} = \underline{\operatorname{Rep}_{B,\mathcal{E}_2}(id)}.$$

<u>Compute:</u>

$$\left[\begin{array}{cc|cc}1&1&1&0\\1&-2&0&1\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc}1&1&1&0\\0&-3&-1&1\end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc}1&1&1&0\\0&1&\frac{1}{3}&-\frac{1}{3}\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc}1&0&\frac{2}{3}&\frac{1}{3}\\0&1&\frac{1}{3}&-\frac{1}{3}\end{array}\right]$$

$$\therefore \underbrace{\begin{bmatrix}1&1\\1&-2\end{bmatrix}}_{Q}^{-1} = \underbrace{\frac{1}{3}\begin{bmatrix}2&1\\1&-1\end{bmatrix}}_{Q^{-1}} \checkmark$$

$$\begin{tikzcd}
V_{\mathcal{E}_2} \arrow[r, "\operatorname{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(L_M)"] \arrow[d, "\operatorname{Rep}_{\mathcal{E}_2,B}(id)"'] & V_{\mathcal{E}_2} \arrow[d, "\operatorname{Rep}_{\mathcal{E}_2,B}(id)"] \\
V_B \arrow[r, "\operatorname{Rep}_{B,B}(L_M)"'] & V_B
\end{tikzcd}$$

$$\begin{tikzcd}
V_{\mathcal{E}_2} \arrow[r, "M"] \arrow[d, "Q"'] & V_{\mathcal{E}_2} \arrow[d, "Q"] \\
V_B \arrow[r, "D"'] \arrow[u, bend right] & V_B
\end{tikzcd}$$

$$D = Q^{-1} M Q$$

$$= \frac{1}{3}\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 8 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 12 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

is a diagonal matrix w/ E-values on diag...

Q: When is M "diagonalizable"?

Defn: Two $n \times n$ matrices A, B are similar when there is an invertible matrix P such that $A = P^{-1} B P$. Matrix M is diagonalizable when there is a diagonal matrix D to which M is similar.

Ex: We just showed $M = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ is similar to $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = D$, so M is diag'ble.

In general, it turns out M $n \times n$ is diagonalizable

* if and only if $\mathbb{R}^n$ has a basis of E-vectors of M.

IDEA: $\underbrace{M} = \underbrace{P^{-1}DP}$

means M and D rep. same transf. $\mathbb{R}^n \to \mathbb{R}^n$ with different bases...

Indeed, $P = \text{Rep}_{B,B'}(id)$...

The E-vectors of M and D are the same...

In particular, for $v \in B'$ $\text{Rep}_{B'}(v) = e_i$

$$D\,\text{Rep}_{B'}(\underline{v}) = De_i = d_{i,i}e_i$$

↖ ith entry on diag of D!

Thus v is an eigenvector for the transformation D represents! Thus B' is a basis for $\mathbb{R}^n$ consisting entirely of E-vectors of L.

Computationally: we can check if M is diag'ble by checking if E-vectors of M contain a basis for $\mathbb{R}^n$...

↳ ① Compute $p_M(\lambda)$.

② Find E-values (via $p_M(\lambda) = 0$)

③ Compute E-vectors For Each λ.

(via solving $(M - \lambda I)\vec{x} = \vec{0}$ and computing a basis of the corresp. space).

④ Check that these bases together form a basis for $\mathbb{R}^n$...

Lem: If M is a matrix w/ distinct E-values λ_1 and λ_2, then the E-spaces U_{λ_1} and U_{λ_2} have only the 0-vector in common.

i.e. any bases for U_{λ_1} and U_{λ_2} are lin. indep. of one another...

$\therefore$ Part ④ becomes:

④' There are n lin. indep E-vectors of M.

$$
\begin{tikzcd}
V_{A} \; \mathcal{E}_3 \arrow[r, "\boxed{\mathrm{Rep}_{A,D}(L)} \leftarrow f"] \arrow[d, "\mathrm{Rep}_{A,A'}(id)"'] & W_{D} \; \mathcal{E}_3 \arrow[d, "\mathrm{Rep}_{D,D'}(id)"] \\
V_{A'} \arrow[r, "\underbrace{\mathrm{Rep}_{A',D'}(L)}_{f}"'] & W_{D'} \\
B \arrow[u] & C \arrow[u]
\end{tikzcd}
$$

$$\mathrm{Rep}_{B,C}(f) = \mathrm{Rep}_{\mathcal{E}_3,C}(id) \cdot \mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(f) \cdot \mathrm{Rep}_{B,\mathcal{E}_3}(id)$$

$$[f]_B^C = \mathrm{Rep}_{B,C}(f).$$